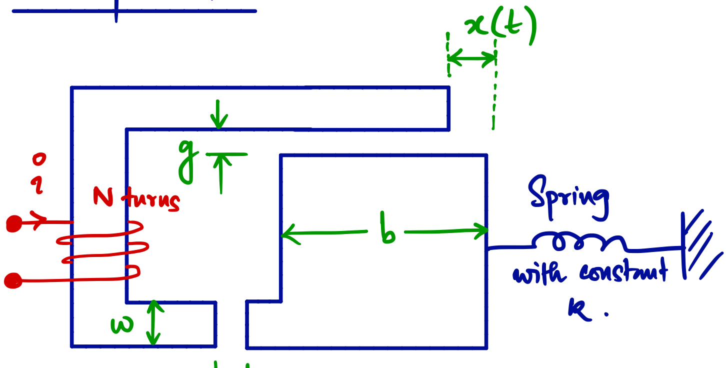
Electromechanical systems.

How does "motion" affect electrical networks?

Recall that emf $v = \dfrac{d\lambda}{dt}$, where $\lambda = N\phi$, and ϕ depends on current and $\underbrace{\text{geometry}}_{\text{Motion can affect the geometry.}}$

Example #1

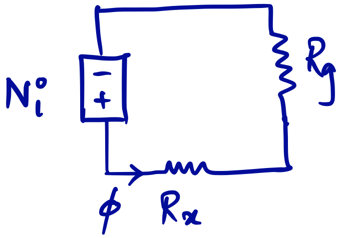


In the adjoining arrangement, the right block can move horizontally.

Compute the emf induced as a function of time.

Assume: $\mu_{core} = \infty$, no fringing, and the depth of each block is w.

Magnetic circuit:



$$R_g = \frac{g}{\mu_0 w (b-x)} \; ;$$

$$R_x = \frac{x}{\mu_0 w^2} .$$

$$\phi = \frac{Ni}{R_g + R_x} = \frac{\mu_0 N i}{\frac{x}{w^2} + \frac{g}{w(b-x)}} = \frac{\mu_0 N w^2 (b-x)}{x(b-x) + gw} \, i .$$

$$\lambda(i, x) = N . \phi = \left(\frac{\mu_0 N^2 w^2 (b-x)}{x(b-x) + gw} \right) . i .$$

λ depends on both x and i. We make this dependency explicit.

$$\lambda(i, x) = L(x) . i, \quad \text{where } L(x) = \frac{\mu_0 N^2 w^2 (b-x)}{x(b-x) + gw} .$$

$$v = \frac{d\lambda}{dt} = L(x) \frac{di}{dt} + \frac{dL(x)}{dt} . i$$

Then, $v(t) = L(x)\dfrac{di}{dt} + \dfrac{dL(x)}{dx}\cdot\left(\dfrac{dx}{dt}\right) i$.

An electrically linear system:

Suppose there are N coils, carrying currents $i_1, i_2, \ldots, i_N$. Also, let there be M variables $x_1, \ldots, x_M$ that define the geometry. Such a system is called electrically linear, if

$$\lambda_j(i_1, i_2, \ldots, i_N, x_1, \ldots, x_M) = \sum_{k=1}^{N} L_{kj}(x_1, \ldots, x_M)\, i_k \quad \ldots$$ linear in currents.

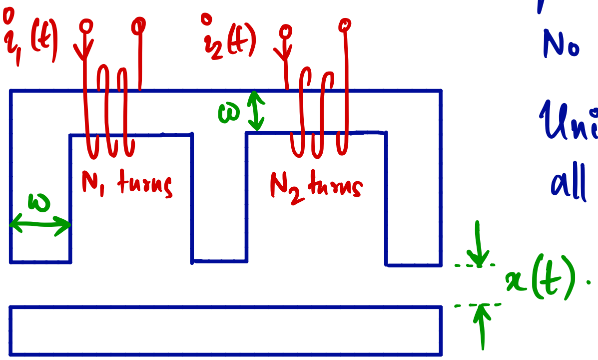
for each $j = 1, \ldots, N$.

Recall that in example #1, $\lambda(i,x) = \left(\dfrac{\mu_0 N^2 w^2 (b-x)}{x(b-x) + gw}\right) \cdot i$

$$= L(x) \cdot i.$$

This is an example of an electrically linear circuit.

Example #2:



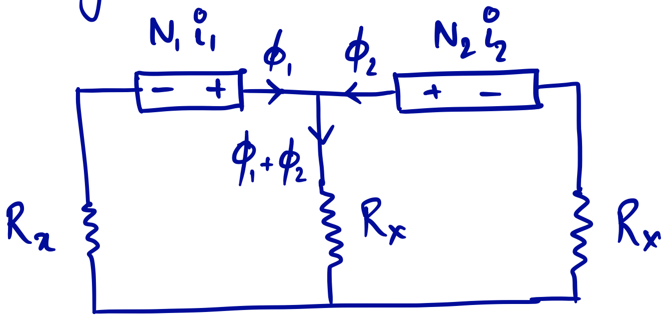
$\mu_{core} = \infty$,

No fringing,

Uniform depth of all structures $= w$.

Consider the arrangement shown here. Compute the voltages/emfs induced across the two coils.

- Magnetic circuit



$$R_x = \frac{x}{\mu_0 w^2}.$$

$$N_1 i_1 = (\phi_1 + \phi_2) R_x + \phi_1 R_x,$$

$$N_2 i_2 = (\phi_1 + \phi_2) R_x + \phi_2 R_x.$$

$$\Rightarrow \begin{pmatrix} 2R_x & R_x \\ R_x & 2R_x \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} N_1 i_1 \\ N_2 i_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{(2R_x)^2 - R_x^2} \begin{pmatrix} 2R_x & -R_x \\ -R_x & 2R_x \end{pmatrix} \begin{pmatrix} N_1 i_1 \\ N_2 i_2 \end{pmatrix}.$$

$$\Rightarrow \phi_1 = \frac{1}{3R_x}\left[2N_1 i_1 - N_2 i_2\right],$$

$$\phi_2 = \frac{1}{3R_x}\left[2N_2 i_2 - N_1 i_1\right].$$

$$\Rightarrow \lambda_1 = N_1 \phi_1 = \frac{2N_1^2}{3R_x} i_1 - \frac{N_1 N_2}{3R_x} i_2,$$

$$\lambda_2 = N_2 \phi_2 = \frac{2N_2^2}{3R_x} i_2 - \frac{N_1 N_2}{3R_x} i_1.$$

Replace R_x by $\frac{x}{\mu_0 w^2}$, and you get

$$\lambda_1(i_1, i_2, x) = \frac{2N_1^2 \mu_0 w^2}{3x} i_1 - \frac{N_1 N_2 \mu_0 w^2}{3x} i_2.$$

This circuit is electrically linear.

$$v_1 = \frac{d\lambda_1}{dt} = \frac{d}{dt}\left[\frac{2N_1^2\mu_0 w^2}{3x} i_1 - \frac{N_1 N_2 \mu_0 w^2}{3x} i_2\right].$$

$$= \left(\frac{2N_1^2\mu_0 w^2}{3}\right)\cdot\left(\frac{1}{x}\frac{di_1}{dt} + i_1\left(\frac{-1}{x^2}\right)\frac{dx}{dt}\right)$$

$$- \left(\frac{N_2 N_2 \mu_0 w^2}{3}\right)\cdot\left(\frac{1}{x}\frac{di_2}{dt} + i_2\left(\frac{-1}{x^2}\right)\frac{dx}{dt}\right).$$

You can calculate v_2 similarly.

Question: Suppose x_1, x_2 describe geometric variables, and i_1, i_2, i_3 describe currents in different coils. Are the following systems electrically linear? λ's denote flux linkages.

$$\begin{cases} \lambda_1 = (3x_1^2 - 4x_2)\, i_1 + 2x_1 x_2\, i_2 + e^{-x_2} i_3, \\ \lambda_2 = 2x_1 x_2\, i_2 + \frac{1}{x_2} i_2 + e^{-x_1} i_3, \\ \lambda_3 = e^{-x_2} i_1 + e^{-x_1} i_2 + (x_1 + x_2)\, i_3. \end{cases}$$

Ans. Yes!

Another example:

$$\lambda_1 = 2x_1 \dot{i}_1 + x_2 \dot{i}_1 \dot{i}_2,$$

$$\lambda_2 = x_2 \dot{i}_1 \dot{i}_2 + (2x_2 + x_1)\dot{i}_1.$$

Ans. No! There is a term $\dot{i}_1 \dot{i}_2$, that is not linear in the currents.